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On The Random Vector Potential Model In Two Dimensions.

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Abstract.
The random vector potential model describes massless fermions coupled to a quenched random gauge field. We study its abelian and non-abelian versions. The abelian version can be completely solved using bosonization. We analyse the non-abelian model using its supersymmetric formulation and show, by a perturbative renormalisation group computation, that it is asymptotically free at large distances. We also show that all the quenched chiral current correlation functions can be computed exactly, without using the replica trick or the supersymmetric formulation, but using an exact expression for the effective action for any sample of the random gauge field. These chiral correlation functions are purely algebraic.

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Two dimensional random systems are much less understood than their pure companions. It is not clearly understood how (if possible) conformal field theory techniques can be applied to random systems. For example, what kind of (probably non-unitary) conformal field theories could describe the infrared fixed points of two dimensional random systems is still an open question.

Besides the familiar study of the effects of disorder on phase transitions [1, 2, 3, 4], there has been a renewed interest in two dimensional random systems in connection with the quantum Hall phase transition [5, 6, 7], or in connection with models of randomly pinned flux lines in superconductors [8].

Obviously, we will not give a complete answer to the question formulated above. However, the models we will study are simple enough, and possess enough symmetries, to allow for exact computations which do not rely on the replica trick or on the supersymmetric formulation. Using this direct computation, we derive some informations on the operator product algebra governing the infrared behavior of these random models. Although our result are only partial results, we hope they provide an information useful enough to start answering the question.

The abelian version of the models was recently introduced for analysing the quantum Hall transition in ref.[9]. The non-abelian version is similar to the model introduced in [10] for describing the effects of impurities interacting with fermions close to the Fermi surface in $2+1$ dimensional systems. However, as explained in the Appendix, it seems that our conclusions are in disagreement with some of those of ref.[10].

1 The model.

i) Definition of the model.

The model describes N massless Dirac fermions minimally coupled to a random non-abelian gauge field with euclidean action : $S=\frac{1}{\pi}\int d^2x\overline{\Psi}(i\partial\!\!\!/ + A\!\!\!/)\Psi$. It is useful to introduce the complex coordinates $z=x+iy$, $\overline{z}=x-iy$ and the components of the fermions $(\overline{\psi}^j_+,\psi^j_+)$ and $(\overline{\psi}_{-;j},\psi_{-;j})$, $j=1,\cdots,N$. The action then becomes :

$$S[A]=\int\frac{d^2z}{\pi}\left(\psi_{-;j}\left(\partial_{\overline{z}}\delta^j_k+A^j_{\overline{z},k}\right)\psi^k_+ +\overline{\psi}_{-;j}\left(\partial_z\delta^j_k+A^j_{z,k}\right)\overline{\psi}^k_+\right) \tag{1}$$

where $A^j_{z,k}=i\sum_a A^a_z(t^a)^j_k$, with $(A^a_z)^*=A^a_{\overline{z}}$, is the gauge field. Here the hermitian matrices t^a form the N-dimensional representation of $U(N)$. We denote by f^{abc} the U(N) structure constants : $[t^a,t^b]=if^{abc}t^c$. The Dirac fermions take values in this N-dimensional representation.

The gauge field is assumed to be a quenched variable with a Gaussian measure :

$$P[A]=\exp\left[-\frac{1}{\sigma}\int\ \frac{d^2z}{\pi}\ \sum_a A^a_zA^a_{\overline{z}}\right] \tag{2}$$

We will be interested in computing the quenched average of the correlation functions of the currents $J^a_\mu=(\overline{\Psi}\gamma_\mu t^a\Psi)$. Explicitely, their components are :

$$\begin{aligned} J^a_z &= \psi_{-;j}(t^a)^j_k\psi^k_+ \\ J^a_{\overline{z}} &= \overline{\psi}_{-;j}(t^a)^j_k\overline{\psi}^k_+ \end{aligned} \tag{3}$$

A convenient way of doing it consists in introducing sources for the currents. This amounts to shift the random field A by an external sources $a_{\rm source}$: $A \to \mathcal{A} = \mathcal{A} + \dashv_{\rm source}$. We are then interested in computing the average of the logarithm of the partition functions with sources :

$$\Gamma[a_{\rm source}] = \overline{W[\mathcal{A}]} = \overline{\log Z[\mathcal{A}]} \tag{4}$$

with $\mathcal{A} = \mathcal{A} + \dashv_{\rm source}$ and

$$Z[A] = e^{W[A]} = \int D\Psi \exp\left(-S[A]\right) = Det[i\partial\!\!\!/ + A\!\!\!/]. \tag{5}$$

$W[A]$ is the generating function for the connected Green function :

$$\langle J^a_\mu(x) J^b_\nu(y)\cdots\rangle^c_A = \pi^{2+\cdots} \frac{\delta W[A]}{\delta A^b_\nu(y)\delta A^a_\mu(x)\cdots}$$

ii) The abelian case.

Before plunging into the non-abelian case, let us first present a simple way of solving the abelian case which corresponds to $N = 1$. This model was introduced in ref.[9] in connection with the quantum Hall transition, and studied there using the replica trick. For $N = 1$, the action (1) can be bosonized. Using the standard rules, $\overline{\Psi}\gamma_\mu\Psi = \epsilon_{\mu\nu}\partial_\nu\Phi$, and $\overline{\Psi} i\partial\!\!\!/ \Psi = \frac{1}{2}(\partial_\mu\Phi)^2$, one finds the bosonic form of the action (1) :

$$S^{N=1}[\Phi, A] = \int \frac{d^2x}{\pi}\left(\frac{1}{2}(\partial_\mu\Phi)^2 + iA_\mu\epsilon_{\mu\nu}\partial_\nu\Phi\right).$$

Since it is a gaussian model, it can be easily solved without using the replica trick. Let us introduce the Hodge decomposition of A_μ : $A_\mu = \partial_\mu\xi + \epsilon_{\mu\nu}\partial_\nu\eta$. The fields ξ and η decouple in the measure (2) :

$$P[\xi;\eta] = \exp\left[-\frac{1}{\sigma}\int \frac{d^2x}{\pi}\ \left((\partial_\mu\xi)^2 + (\partial_\mu\eta)^2\right)\right] \tag{6}$$

The action $S^{N=1}[\Phi, A]$ is independent of ξ and therefore the field ξ is irrelevant. This fact was expected since the field ξ represents a pure gauge whereas the physically relevent quantity is the field strengh $F = \epsilon_{\mu\nu}\partial_\mu A_\nu = (\partial_\mu\partial_\mu)\eta$. Moreover, the field η can be absorbed into a translation of Φ :

$$S^{N=1}[\Phi, A_\mu = \epsilon_{\mu\nu}\partial_\nu\eta] = S^{N=1}[\Phi + i\eta, A = 0] + \frac{1}{2}\int \frac{d^2x}{\pi}(\partial_\mu\eta)^2. \tag{7}$$

This equation encodes the anomalous transformation law of the determinant $Det[i\partial\!\!\!/ + A\!\!\!/]$ under a chiral gauge transformation of the abelian gauge field A :

$$\frac{Det[i\partial\!\!\!/ + A\!\!\!/]}{Det[i\partial\!\!\!/]} = \exp\left[-\frac{1}{2}\int \frac{d^2x}{\pi}\ (\partial_\mu\eta)^2\right]$$

The fact that the field η can be absorbed into a shift of Φ does not mean that the quenched correlation functions are identical to those in the pure system. Using eq.(7), we have

$$\langle\prod_n e^{i\alpha_n\Phi(x_n)}\rangle_{A_\mu=\epsilon_{\mu\nu}\partial_\nu\eta} = e^{\sum_n \alpha_n\eta(x_n)}\quad \langle\prod_n e^{i\alpha_n\Phi(x_n)}\rangle_{A=0}.$$

It factorizes into the product of the correlation functions in the pure system times a simple function of the impurities. However, the average of this function is not irrelevant since the variables η have long-range correlations : $\overline{\eta(x)\eta(y)} = -\pi\sigma(\partial_\mu\partial_\mu)^{-1}(x,y) = -\sigma \log|x-y|$. In particular, it changes the values of the critical exponents. The dimensions Δ of the vertex operators $\exp(i\alpha\Phi(x))$ in the quenched theory are :

$$\Delta^{\alpha}_{quenched} = \Delta^{\alpha}_{pure} - \sigma\alpha^2.$$

Averages of product of correlation functions can be computed similarly. One has :

$$\begin{aligned}\overline{\langle\prod_n e^{i\alpha_n\Phi(x_n)}\rangle_A \cdots \langle\prod_m e^{i\beta_n\Phi(y_m)}\rangle_A} &= \langle\langle e^{\sum_n \alpha_n\eta(x_n)+\cdots+\sum_m \beta_m\eta(y_m)}\rangle\rangle \times \\ &\times \langle\prod_n e^{i\alpha_n\Phi(x_n)}\rangle_{A=0} \cdots \langle\prod_m e^{i\beta_n\Phi(y_m)}\rangle_{A=0}\end{aligned}$$

where $\langle\langle\cdots\rangle\rangle$ refers to the η-correlation functions with the free field measure (6). Clearly, conformal invariance is unbroken in the random abelian case. More details will be described elsewhere [12].

This way of solving this very simple model is closely related to the change of variables used in ref.[11]. What we are going to present is the non-abelian analogue of this construction. In the following, we will restrict ourself to the $SU(N)$ sector of $U(N)$.

iii) Symmetries and Ward identities.

The currents $J^a_\mu = (\overline{\Psi}\gamma_\mu t^a\Psi)$ are conserved in the pure system, i.e $\langle(\partial_\mu J^a_\mu)\ \mathcal{O}\rangle_{A=0} = 0$. It is well known in conformal field theory that in the pure system these currents represent a Kac-Moody algebra, alias a current algebra, of level one [13]. This is non longer true in the presence of impurities. However, the measure $P[A]$, eq.(2), is invariant under a global $SU(N)$ rotation : $A_\mu \to UA_\mu U^{-1}$ with $U \in SU(N)$. The Noether current associated to this symmetry is A_μ, which is therefore conserved inside any quenched correlation functions. I.e. $\overline{(\partial_\mu A_\mu)\ \langle\mathcal{O}_1\rangle_A \cdots \langle\mathcal{O}_M\rangle_A} = 0$. By integration by part, this is equivalent to the conservation law for the currents $\mathcal{J}^a_\mu$ defined by :

$$\mathcal{J}^a_\mu = \pi\frac{\delta}{\delta A^a_\mu}. \tag{8}$$

Explicitely, the Ward identities are :

$$\overline{\partial_\mu\ \left(\mathcal{J}^a_\mu\left[\langle\mathcal{O}_1\rangle_A \cdots \langle\mathcal{O}_M\rangle_A\right]\right)} = 0. \tag{9}$$

Equivalently, by computing the action of $\mathcal{J}^a_\mu$ on the expectation values, we get :

$$\sum_j \overline{\langle\mathcal{O}_1\rangle_A \cdots \left[\langle\partial_\mu J^a_\mu \mathcal{O}_j\rangle_A - \langle\partial_\mu J^a_\mu\rangle_A\langle\mathcal{O}_j\rangle_A\right] \cdots \langle\mathcal{O}_M\rangle_A} = 0.$$

Therefore, in the quenched theory the conserved currents are not J^a_μ but their insertions in the connected Green functions. This point can also be recovered using the replica formalism. (There also exist Ward identites for chiral gauge transformations : $A_z \to UA_zU^{-1}$ and $A_{\overline{z}} \to VA_{\overline{z}}V^{-1}$). As usual, the Ward identites (9) are up to contact terms which encode the transformation law of the fields.

Usually, conformal field theories with a global $SU(N)$ symmetry group possess two chiral sets of $SU(N)$ conserved currents satisfying a Kac-Moody algebra. So, it seems reasonable

to expect that the field theory describing the large distance behavior of this quenched model should possess two sets of currents. We will verify this point in the following section while computing the chiral quenched correlation function. We will actually obtain a little more since we will find that the *chiral* correlation functions are purely algebraic and do not present any crossover between their ultraviolet and infrared behaviors.

2 The supersymmetric approach.

A standard approach for studying this random model would be to use the replica trick or the supersymmetric formalism. In this section, we use a perturbative renormalization group computation to derive the infrared behavior of a few averaged correlation functions.

i) The supersymmetric action.

Since the partition function is a determinant, we choose to use the supersymmetric formalism. Therefore, we introduce auxiliary bosonic fields, denoted (b_j, c^j), in order to represent the inverse of the partition function (5) :

$$\frac{1}{Z[A]} = \int DbDc \quad \exp(-S_{aux}[A])$$

with

$$S_{aux}[A] = \int \frac{d^2z}{\pi} \left(b_j \left(\partial_{\overline{z}} \delta^j_k + A^j_{\overline{z},k}\right) c^k + \overline{b}_j \left(\partial_z \delta^j_k + A^j_{z,k}\right) \overline{c}^k \right) \tag{10}$$

In order to calculate averages of products of correlation functions, we need to introduce as many copies of the fermions and the auxiliary bosons as necessary. We denote them by $(\overline{\Psi}^\alpha, \Psi^\alpha)$ and (b^α, c^α). Their dynamic is governed by the action $S_{tot}[A]$ with :

$$S_{tot}[A] = \sum_\alpha \left(S^\alpha[A] + S^\alpha_{aux}[A]\right). \tag{11}$$

The fields are only coupled through the impurities. After integration over A with the measure (2) we obtain the effective action :

$$S_{eff} = \sum_\alpha \left(S^\alpha[A=0] + S^\alpha_{aux}[A=0]\right) - \sigma \sum_a \int \frac{d^2z}{\pi} (\sum_\alpha H^{a;\alpha}_z)\ (\sum_\alpha H^{a;\alpha}_{\overline{z}}). \tag{12}$$

where H^a_z and $H^a_{\overline{z}}$ denote the total $SU(N)$ currents :

$$\begin{aligned} H^a_z &= \psi_{-;j}(t^a)^j_k \psi^k_+ + b_j (t^a)^j_k c^k \\ H^a_{\overline{z}} &= \overline{\psi}_{-;j}(t^a)^j_k \overline{\psi}^k_+ + \overline{b}_j (t^a)^j_k \overline{c}^k. \end{aligned}$$

The action S_{eff} is explicitely supersymmetric. More precisely, the action $S_{tot}[A]$ is supersymmetric before any integration over A. There are two supercharges, Q and $\overline{Q}$, acting separately on the left and right sectors. The charges Q, acting only on the left movers, is defined by :

$$\begin{aligned} Q(\psi_-) = 0 \quad &; \quad Q(b) = \psi_-, \\ Q(\psi_+) = c \quad &; \quad Q(c) = 0. \end{aligned}$$

Q acts trivially on A and on the right sector. Clearly, it satisfies $Q^2 = 0$. One verifies that $S_{tot}[A]$ is susy invariant : $Q(S_{tot}[A]) = \overline{Q}(S_{tot}[A]) = 0$. One also verifies that the total current H^a_z and $H^a_{\overline{z}}$ are respectively Q and $\overline{Q}$ closed. This means that there exist two local fields K^a_z and $K^a_{\overline{z}}$ such that :

$$H^a_z = Q(K^a_z) \quad and \quad H^a_{\overline{z}} = \overline{Q}(K^a_{\overline{z}}).$$

This property ensures that the total partition function, obtained by integrating over all the fields (the fermions plus the bosons), is independent of A as it should be.

There exists an intriguing similarity between the above construction and conformal topological field theory in two dimensions. In the latter, one uses a twisted version of a $N = 2$ supersymmetry to construct the theory such that the stress tensor is susy closed. The closure of the stress tensor ensures that the partition function of the topological theory is independant of the background metric. In the random model, we can extend the supersymmetric algebra we just described such that the new algebra possesses two supersymmetric charges. The fact that the current is susy closed is the analogue of the fact that the stress tensor is susy closed. It seems natural to wonder if techniques of conformal topological field theories cannot be translated into new methods of analysis of two dimensional random systems.

ii) A renormalisation group computation.

The only coupling constant which can be renormalized is σ. As usual, the one-loop β and γ-functions are encoded in the operator product expansions of the fields in the unperturbed theory. These can be read off from the action $S_{tot}[A = 0]$. We have :

$$\psi^k_+(z)\ \psi_{-;j}(w) \sim \frac{\delta^k_j}{(z-w)} + regular,$$

$$c^k(z)\ b_j(w) \sim \frac{\delta^k_j}{(z-w)} + regular.$$

The total currents H^a_z satisfy a Kac-Moody algebra with zero central charge :

$$H^a_z(z)\ H^b_z(w) \sim \frac{if^{abc}}{z-w} H^c_z(w) + reg.$$

There is an exact compensation between the central charges associated to the fermions and to the auxiliary bosons.

The currents H^a_z have dimension one; therefore, the β function vanishes at tree level. At one loop, one finds :

$$\dot{\sigma} \equiv \beta(\sigma) = -C_G\ \sigma^2 + \mathcal{O}(\sigma^3), \tag{13}$$

where $C_G > 0$ is the Casimir in the adjoint representation, $f^{abc} f^{dbc} = C_G \delta^{ad}$. The sign in eq.(13) is important; it tells us that the model is asymptotically free in the infrared regime. We have :

$$\sigma(R) = \frac{\sigma_0}{1 + \sigma_0 C_G \log(R/a)},$$

with $\sigma_0 > 0$ the value of the coupling constant at the lattice cut-off a. Since the model is asymptotically free in the infrared, we can use the renormalization group to evaluate the large distance behavior of the two-point correlation functions using the usual formula :

$$\langle \Phi(R)\Phi(0)\rangle_{\sigma_0} = \langle \Phi(a)\Phi(0)\rangle_{\sigma(R)} \exp\left(-2 \int_{\sigma_0}^{\sigma(R)} d\overline{\sigma} \frac{\gamma_\Phi(\overline{\sigma})}{\beta(\overline{\sigma})} \right), \tag{14}$$

where γ_Φ is the γ-function for Φ.

Let us first compute the asymptotic of the chiral two-point function $\overline{\langle J^a_z(z) J^b_z(0)\rangle_A}$. To evaluate it we only need one copy of the fermion and their supersymmetric partner. We find that the γ-function of J^a_z is $\gamma_J(\sigma) = 1 + \mathcal{O}(\sigma^2)$; i.e. there is no one-loop correction even though J^a_μ are not conserved. Therefore, at large distances we have :

$$\overline{\langle J^a_z(z) J^b_z(0)\rangle_A} \sim \frac{\delta^{ab}}{z^2}. \tag{15}$$

In particular, there are no logarithmic correction.

Consider now the average of the product of two such two-point functions but of opposite chiralities, i.e. $\overline{\langle J^a_z(z) J^b_z(0)\rangle_A \langle J^{\overline{a}}_{\overline{z}}(z) J^{\overline{b}}_{\overline{z}}(0)\rangle_A}$. To compute it we need two copies of the fermions. These quenched averages are then represented by the two-point functions of the operators $\mathcal{O}^{a\overline{a}}_{12} = J^{a;(1)}_z(z) J^{\overline{a};(2)}_{\overline{z}}(z)$, products of currents in the different copies. For these operators the γ-functions have one-loop corrections. There is a mixing between the operators $\mathcal{O}^{a\overline{a}}_{12}$ with different indices :

$$\gamma^{a\overline{a}}_{b\overline{b}} = 2\delta^a_b \delta^{\overline{a}}_{\overline{b}} + 2\sigma f^{da}_b f^{d\overline{a}}_{\overline{b}} + \mathcal{O}(\sigma^2).$$

In particular the $SU(N)$ scalar operator $\mathcal{O} = \sum_a J^{a;(1)}_z J^{a;(2)}_{\overline{z}}$ does not mix with the others, and $\gamma_{\mathcal{O}} = 2 + 2C_G + \mathcal{O}(\sigma^2)$. Therefore,

$$\sum_{a,b} \overline{\langle J^a_z(z) J^b_z(0)\rangle_A \langle J^a_{\overline{z}}(z) J^b_{\overline{z}}(0)\rangle_A} \sim \left(\frac{a^2}{z\overline{z}}\right)^2 (\log|z\overline{z}|)^{-4} \tag{16}$$

Notice the logarithmic correction which breaks the self-averaging property. It is remarkable that the logarithms appear in the averages of products of correlations of opposite chiralities, eq.(16), but not in the averages of the chiral correlations, eq.(15).

Infrared asymptotics of averages of higher products of current two-point functions, or of fermion correlations, can be computed similarly. In particular, we find logarithmic corrections for the quenched two-point function $\overline{\langle \epsilon(R)\epsilon(0)\rangle_A}$ for the energy operator $\epsilon = \overline{\Psi}\Psi$:

$$\overline{\langle \epsilon(z)\epsilon(0)\rangle_A} \sim \left(\frac{a^2}{z\overline{z}}\right) (\log|z\overline{z}|)^{-4C_V/C_G}.$$

where C_V is the $SU(N)$ Casimir in its N-dimensional representation.

3 The quenched chiral correlation functions.

In this section, we describe how some of the quenched correlation functions can be directly computed without using the replica trick or the supersymmetric formulation. This relies on the Polyakov-Wiegman (PW) formula [14] for the effective action $W[A]$, eq.(4). Once this formula has been recalled, the computations are rather simple.

i) The Polyakov-Wiegman formula.

The PW formula relies on the fact that in two dimensions the effective action can be exactly computed by integrating its anomalous transformation under a chiral gauge transformation. This is the non-abelian analogue of eq.(7). Indeed, let $G_\mu = \sum_a t^a G^a_\mu[A]$ be the

generating functions of the connected current Green function in the pure system :

$$G^a_\mu[A] \equiv \langle J^a_\mu \rangle_A = \pi \frac{\delta W[A]}{\delta A^a_\mu}. \tag{17}$$

They satisfy the anomalous Ward identities [14],

$$\begin{aligned} \partial_z G_{\overline{z}} + \partial_{\overline{z}} G_z + [A_z, G_{\overline{z}}] + [A_{\overline{z}}, G_z] &= 0, \\ \partial_z G_{\overline{z}} - \partial_{\overline{z}} G_z + [A_z, G_{\overline{z}}] - [A_{\overline{z}}, G_z] &= 2F_{z\overline{z}}[A], \end{aligned} \tag{18}$$

with $F_{z\overline{z}}[A] = \partial_z A_{\overline{z}} - \partial_{\overline{z}} A_z + [A_z, A_{\overline{z}}]$. Eqs.(18) completely specify $G_\mu[A]$:

$$G_z[A] = A_z - \frac{1}{\partial_{\overline{z}} + ad.A_{\overline{z}}} \partial_z A_{\overline{z}} \tag{19}$$

$$G_{\overline{z}}[A] = A_{\overline{z}} - \frac{1}{\partial_z + ad.A_z} \partial_{\overline{z}} A_z \tag{20}$$

Here $ad.A_z$ denoted the adjoint action of A_z. Notice that $G_z[A]$ is local in A_z but non-local in $A_{\overline{z}}$.

ii) Explicit formula for the chiral correlation functions.

Knowing explicitely the correlation functions for any impurity sample, it is a priori possible to take the quenched average. But let us first concentrate on the average of the chiral correlation functions involving only currents of the same chirality; e.g. involving only J^a_z. Consider first the average of products of one-point functions (17). Since the quenched average is defined by $\overline{A^a_z(x) A^b_{\overline{z}}(y)} = \sigma\pi\delta^{ab}\delta^{(2)}(x-y)$ and since $\langle J^a_z \rangle_A$ are linear in A_z, these quenched correlations can be computed using the Wick theorem applied on A. For exemple, the two-point and three-point functions are :

$$\begin{aligned} \overline{\langle J^a_z(z_1)\rangle_A \langle J^b_z(z_2)\rangle_A} &= 2\sigma\pi\ \delta^{ab} \left(\frac{1}{\partial_{\overline{z}}}\ \partial_z\right)_{(z_1,z_2)} = \frac{2\sigma\delta^{ab}}{(z_1-z_2)^2} \\ \overline{\langle J^a_z(z_1)\rangle_A \langle J^b_z(z_2)\rangle_A \langle J^c_z(z_3)\rangle_A} &= i(\pi\sigma)^2 f^{abc} \left(\frac{1}{\partial_{\overline{z}}}\ \partial_z\right)_{(z_1,z_2)} \left[\left(\frac{1}{\partial_{\overline{z}}}\right)_{(z_3,z_1)} - \left(\frac{1}{\partial_{\overline{z}}}\right)_{(z_3,z_2)}\right] \\ &\quad + (cyclic\ \ permutation) \\ &= 3\sigma^2 \frac{i f^{abc}}{(z_1-z_2)(z_1-z_3)(z_2-z_3)} \end{aligned}$$

More generally, the average of products of one-point functions is the sum of connected correlations which can be expressed in terms of the correlation functions of the pure system :

$$\left[\overline{\langle J^{a_1}_z(z_1)\rangle_A \cdots \langle J^{a_M}_z(z_M)\rangle_A}\right]^{connected} = M\sigma^{M-1}\langle J^{a_1}_z(z_1)\cdots J^{a_M}_z(z_M)\rangle_0 \tag{21}$$

Here, $\langle\cdots\rangle_0$ denote the pure correlation functions. They are known exactly [15]. The relation between the quenched correlations and their connected parts is the usual one.

More interesting is the average of products of correlations with insertion of the conserved currents (8) since they encode the underlying symmetry algebra. We find :

$$\begin{aligned} &\overline{\mathcal{J}^{n_1}_z(z_1)\cdots\mathcal{J}^{n_M}_z(z_M)\langle J^{a_1}_z(w_1)J^{b_1}_z(\xi_1)\cdots\rangle^c_A \cdots \langle J^{a_P}_z(w_P)J^{b_P}_z(\xi_P)\cdots\rangle^c_A} \\ = \ &\langle J^{n_1}_z(z_1)\cdots J^{n_M}_z(z_M) J^{a_1}_z(w_1)J^{b_1}_z(\xi_1)\cdots\rangle_0 \cdots \langle J^{a_P}_z(w_P)J^{b_P}_z(\xi_P)\cdots\rangle_0 \end{aligned}$$

$$
\begin{aligned}
&+ \quad \sum_{j=1}^{M} \langle J_z^{n_1}(z_1) \cdots \widehat{J_z^{n_j}(z_j)} \cdots J_z^{n_M}(z_M) J_z^{a_1}(w_1) J_z^{b_1}(\xi_1) \cdots \rangle_0 \times \\
&\qquad\qquad \times \langle J_z^{n_j}(z_j) J_z^{a_2}(w_2) J_z^{b_2}(\xi_2) \cdots \rangle_0 \cdots \langle J_z^{a_P}(w_P) J_z^{b_P}(\xi_P) \cdots \rangle_0 \\
&+ \quad \cdots \\
&+ \quad \langle J_z^{a_1}(w_1) J_z^{b_1}(\xi_1) \cdots \rangle_0 \cdots \langle J_z^{n_1}(z_1) \cdots J_z^{n_M}(z_M) J_z^{a_P}(w_P) J_z^{b_P}(\xi_P) \cdots \rangle_0
\end{aligned} \tag{22}
$$

The hatted fields have to be omitted. Here, we assumed that there is no insertion of one-point functions. The formula (22) is actually simpler in words : it is obtained by distributing the currents $J_z^{n_1}(z_1), \cdots, J_z^{n_M}(z_M)$ among the pure correlators in all possible way, each counted only once.

Notice that all the chiral quenched correlation functions are purely algebraic, without any logarithmic correction, in agreement with the renormalization group computation. So, conformal invariance does not seem to be broken in the random model.

From eq.(22) we read the operator product expansion of the fields. The currents satisfy :

$$
\mathcal{J}_z^{n_1}(z_1)\mathcal{J}_z^{n_2}(z_2) = \frac{\delta^{n_1 n_2}}{(z_1 - z_2)^2} + \frac{i f^{n_1 n_2 n_3}}{z_1 - z_2} \mathcal{J}^{n_3}(z_2) + reg. \tag{23}
$$

Therefore, the quenched conserved currents satisfy the commutation relations of a Kac-Moody algebra, exactly as the currents in the pure system do.

The operator product expansion between the conserved currents and the correlators $\langle J_z^{a_1}(w_1) J_z^{a_2}(w_2) \cdots \rangle_A$ are :

$$
\begin{aligned}
\mathcal{J}_z^{n}(z) \langle J_z^{a_1}(w_1) J_z^{a_2}(w_2) \cdots \rangle_A^c \;&=\; \sum_j \frac{\delta^{n a_j}}{(z - w_j)^2} \langle J_z^{a_1}(w_1) \cdots \widehat{J_z^{a_j}(w_j)} \cdots \rangle_A^c \\
&+\; \sum_j \frac{i f^{n a_j a'_j}}{z - w_j} \langle J_z^{a_1}(w_1) \cdots J_z^{a'_j}(w_j) \cdots \rangle_A^c + reg.
\end{aligned} \tag{24}
$$

These operator product expansions are unusual in conformal field theory with Kac-Moody symmetry. In particular, they imply that the fields $\langle J_z^{a_1}(w_1) J_z^{a_2}(w_2) \cdots \rangle_A^c$ are *not* associated to highest weight vector representations. We don't know to which category of representations they correspond to.

Contrary to the chiral quenched correlation functions which are easy to compute, the averages of correlation functions involving fields of opposite chiralities are difficult to evaluate. This is due to the fact that the generating functions $G_z[A]$ and $G_{\overline{z}}[A]$ are non-local in $A_{\overline{z}}$ and A_z respectively. A naive perturbative expansion is spoilled by untractable divergences. This could have been anticipated in view of the logarithmic corrections present in eq.(16). The occurence of these logarithms probably means that the gluing of the left and right sectors is more subtle than usual. In particular, one has to learn how to deal with an infinite number of primary fields. We don't know any good algebraic way of doing it correctly in the non-abelian case, but we hope that the remarks presented in this note will be useful to find the answer.

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While finishing writing this note, a related paper appears on the bulletin board [16].

4 Appendix.

In this appendix, we explain why our results seem to be in contradiction with some of those presented in ref.[10]. The approach used in this paper is based on the replica trick. Thus, the authors introduce n copies of the N fermions, forming alltogether Nn Dirac fermions. These fermions form a representation of level one of the $SU(Nn)$ Kac-Moody algebra. After averaging the random potential, these Nn free massless Dirac fermions are coupled by a current-current interaction :

$$S_{int} = \sigma \int \frac{d^2z}{\pi} (\sum_p J_\mu^{a;p})(\sum_p J_\mu^{a;p}) \tag{25}$$

where the currents $J_\mu^{a;p}$ are the $SU(N)$ currents in the p^{th} replica. Not all the degrees of freedom are coupled by this interaction since only the diagonal sum of the $SU(N)$ currents interacts. The authors of ref.[10] argue that the interaction opens a gap and that some of the modes become massive. More precisely, let us decompose the $SU(Nn)_{k=1}$ representation as $SU(Nn)_{k=1} = SU(N)_{k=n} \times SU(n)_{k=N}$. They argue that the modes corresponding to $SU(N)_{k=n}$ become massive and therefore that the infrared behavior is described by the $SU(n)_{k=N}$ conformal field theory. In particular, this would implies that the correlation functions of the diagonal $SU(N)$ currents $(\sum_p J_\mu^{a;p})$ decrease exponentially at infinity. But, since these correlation functions at $n = 0$ give the quenched averages of the connected correlation functions of the currents, the latter statement is in contradiction with the result we obtained using the Polyakov-Wiegmann formula.

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